LINEAR PROGRAMMING

"THE MATHEMATICAL EXPERIENCE OF THE STUDENT IS INCOMPLETE IF HE NEVER HAD THE OPPORTUNITY TO SOLVE A PROBLEM INVENTED BY HIMSELF" G.POLYA

https://youtu.be/Uo6aRV-mbeg

https://youtu.be/qQFAvPF2OSI

NEW WORDS RELATED TO LPP

- LPP → $Linear\ programming\ problems$
- Optimal Value → Maximum or Minimum Value
- Objective Function → Maximum or Minimum function
- Constrains → A set of Linear inequalities in the problem.
- Decision Variables → Non negative Variables Like x and y
- Feasible Region → Solution region (common shaded region)
- Feasible solution → Points within and on the boundary of the feasible solutions.
- Optimal solution → Any point in the feasible region that gives the max/min value of the objective function.

INTRODUCTION

- **In this chapter we shall apply the system of linear inequalities/equations to solve some real life problems.**
- **Problems which seek to maximize the profit or minimize the cost , form a general class of problems called optimization problems.**
- **A special but a very important class of optimization problems is Linear Programming problems.**

Q1

- A furniture dealer deals in only two items–tables and chairs. He has Rs 50,000 to invest and has storage space of at most 60 pieces. A table costs Rs 2500 and a chair Rs 500. He estimates that from the sale of one table, he can make a profit of Rs 250 and that from the sale of one chair a profit of Rs 75. He wants to know how many tables and chairs he should buy from the available money so as to maximise his total profit, assuming that he can sell all the items which he buys.

ANALYSING THE QUESTION & FORMULATION

Mathematical formulation of the problem

INVESTMENT-50000; STORAGE-60; CP/TAB-2500;CP/CHAIR- 500 PROFIT/TABLE-250;PROFIT/CHAIR-75

- Let x be the number of tables and y be the number of chairs
- $x \geq 0 : y \geq 0$ ....**non negative constrains**
- $$\left.\begin{array}{l} 2500x + 500\,y \leq 50000 \rightarrow 5x + y \leq 100 \;---(1) \\ x + y \leq 60 \;-----------------------(2) \end{array}\right\} \textbf{constraints}$$
- The dealer wants to invest in such a way so as to maximize his profit say, Z

  $Z = 250\,x + 75\,y$ ....**Objective function**

GRAPHICAL REPRESENTATION

- GRAPH

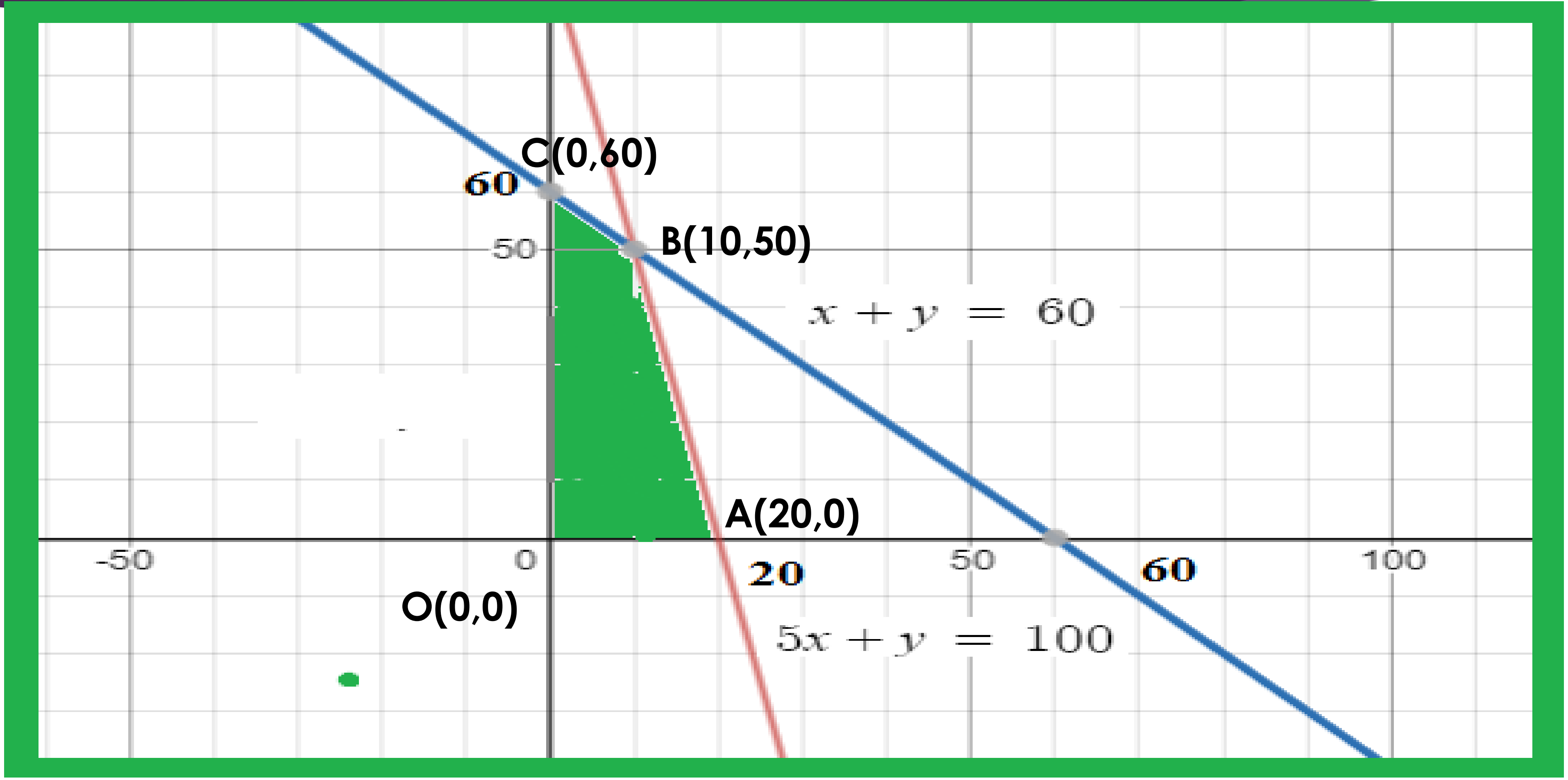


Table and conclusion

CORNER POINTS	Z = 250 x + 75 y
O(0,0)	Rs 0
A(20,0)	Rs5000
B(10,50)	Rs6250→ *maximum*
C(0,60)	Rs4500

Hence, He has to buy 10 tables and 50 chairs to make the profit maximum Rs 6250

Conclusion

- Hence, He has to buy 10 tables and 50 chairs to make the profit maximum Rs 6250

Q 2

- Question 2:

One kind of cake requires 200g flour and 25g of fat, and another kind of cake requires100g of flour and 50g of fat. Find the maximum number of cakes which can be made from 5 kg of flour and 1 kg of fat assuming that there is no shortage of the other ingredients used in making the cakes?

TABLE FOR ANALYSING THE QUESTION

- Let there be x cakes of first kind and y cakes of second kind.

The given information can be complied in a table as follows.

	FIRST TYPE CAKE x	SECOND TYPE CAKE y	Availability
FLOUR(g)	200g	100g	≤ 5000g
FAT (g)	25g	50g	≤1000g

FORMULATION

- Total numbers of cakes, Z, that can be made are, Z = x + y,
- The mathematical formulation of the given problem is Maximize

  Z = x + y ....objective function

  subject to the Constrains, $200x + 100y \leq 5000 \rightarrow 2x + y \leq 50$.........(1)

  $25x + 50y \leq 1000 \rightarrow x + 2y \leq 40$ ....(2)

  $x \geq 0 , y \geq 0$ ....(3)(non negative constrains)

GRAPHICAL REPRESENTATION

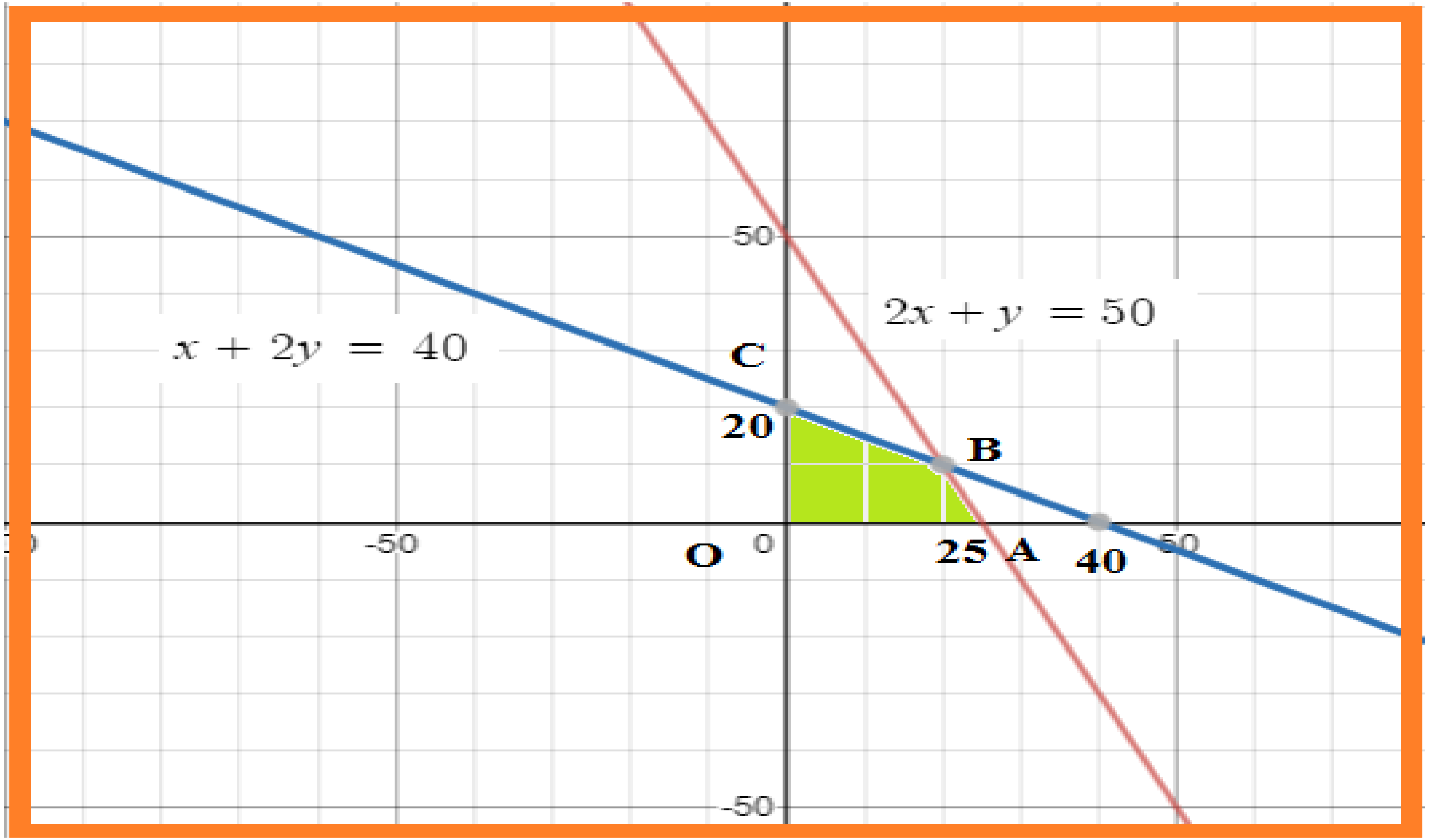


Table and conclusion

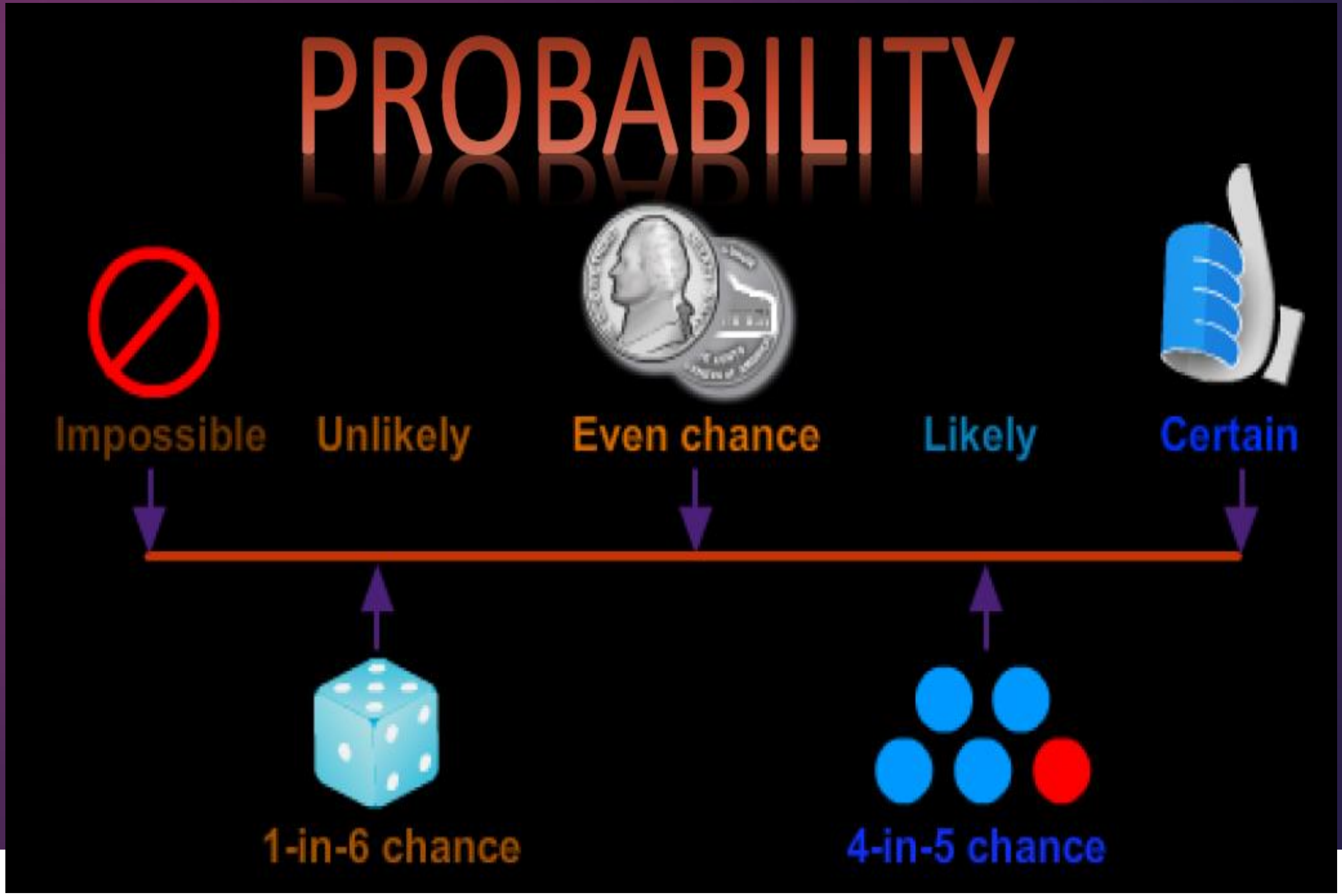
CORNER POINTS	Z = x + y
O(0,0)	0
A (25,0)	25
B (20,10)	30 $\leftarrow Maximum$
C (0,20)	20

Thus, the maximum numbers of cakes that can be made are 30 (20 of one kind and 10 of the other kind).

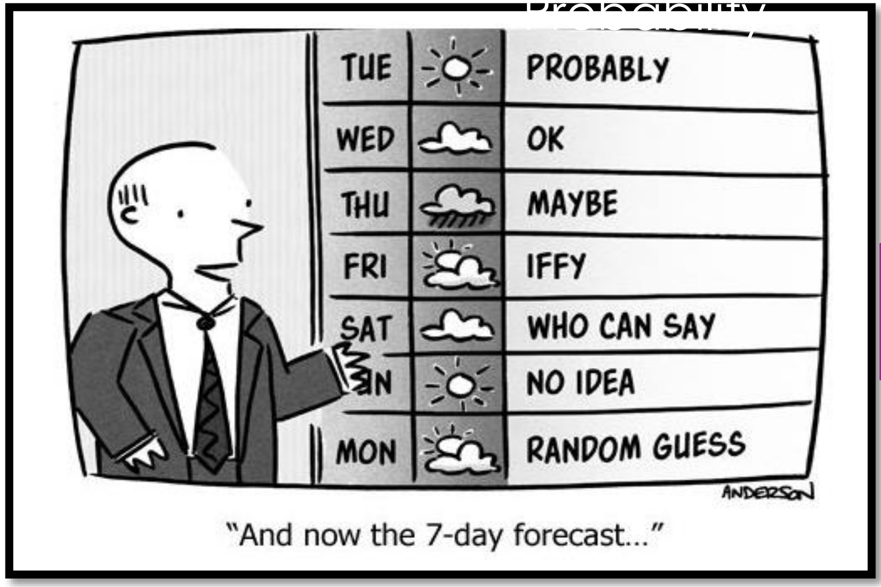
PROBABILITY

CLASS :XII
CHAPTER-13
MODULE-1

LIFE IS A SCHOOL OF PROBABILITY.



Real Life Examples Of Probability



1. Weather Forecasting

2. Batting Average in Cricket

3.Insurance

Probability helps in analyzing the best plan of insurance which suits you and your family the most

4.Lottery Tickets

Winning or losing a lottery is one of the most interesting examples of probability

Classical Probability

Classical probability is based on the assumption that the outcomes of an experiment are *equally likely*. Using the classical viewpoint, the probability of an event happening is computed by dividing the number of favorable outcomes by the number of possible outcomes:

CLASSICAL PROBABILITY

$$\text{Probability of an event} = \frac{\text{Number of favorable outcomes}}{\text{Total number of possible outcomes}}$$

EXAMPLE

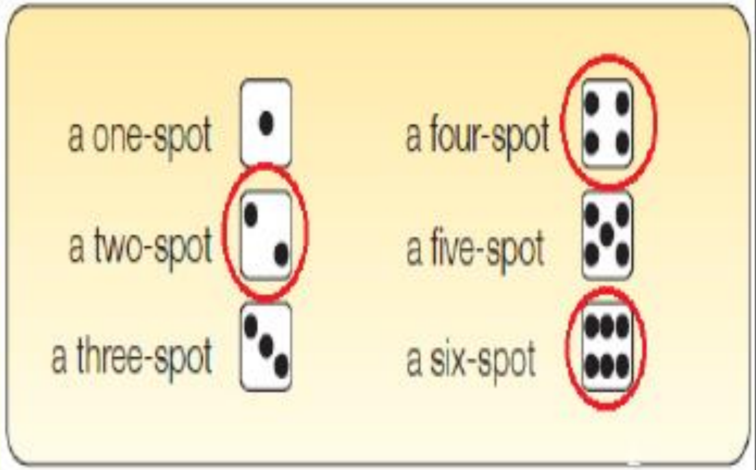
Consider an experiment of rolling a six-sided die. What is the probability of the event "an even number of spots appear face up"?

SOLUTION

The possible outcomes are:

$\{ 1, 2, 3, 4, 5, 6, \}$

There are three "favorable" outcomes (a two, a four, and a six)



$$\text{Probability of an even number} = \frac{3}{6} \begin{matrix} \leftarrow \\ \leftarrow \end{matrix} \frac{\text{Number of favorable outcomes}}{\text{Total number of possible outcomes}}$$

$$= \frac{1}{2}$$

Empirical Probability

EMPIRICAL PROBABILITY The probability of an event happening is the fraction of the time similar events happened in the past.

$$\text{Empirical probability} = \frac{\text{Number of times the event occurs}}{\text{Total number of observations}}$$

EXAMPLE

On February 1, 2003, the Space Shuttle *Columbia* exploded. This was the second disaster in 113 space missions for NASA. On the basis of this information, what is the probability that a future mission is successfully completed?

SOLUTION

We use letters or numbers to simplify the equations. *P* stands for probability and *A* represents the event of a successful mission. In this case, *P*(*A*) stands for the probability a future mission is successfully completed.

$$\text{Probability of a successful flight} = \frac{\text{Number of successful flights}}{\text{Total number of flights}}$$

$$P(A) = \frac{111}{113} = .98$$

GENERAL RULE OF ADDITION $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

MUTUALLY EXCLUSIVE The occurrence of one event means that none of the other events can occur at the same time.

Eg. 1 . Selecting a male or female.

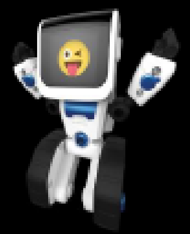
Eg. 2. Selecting a King or a queen card.

Eg. 3. Selecting Odd or even numbers in a throw of a dice.

Eg. 4. Selecting a diamond or a heart card.

Eg. 5. Selecting a head or tail from a single throw of a coin.

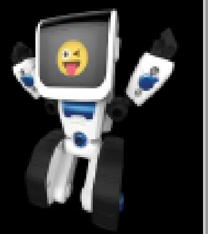
In case of mutually exclusive events the rule of addition can be simply be taken as P(A or B) = P(A) + P(B) $\because A \cap B = \emptyset$



combined or compound events

In probability, combined or compound events are when two or more experiments are conducted together. Also called multiple events.

An event is one or more outcomes from the set of all possible outcomes in a probability experiment.



two types of combined events

independent events

Independent events do not affect or are not affected by another event or events.

They are the opposite of dependent events in which the outcome of one event affects or is affected by the outcome of another event or events.

probability

If a students in a class each take turns with a spinner, each person has the exactly same chance of getting purple as everyone else, that is, a probability of 1 in 5.

Each spin is an independent event, not affected by any previous events.

dependent events

Dependent events are those in which the outcome of one event affects or is affected by the outcome of another event or events.

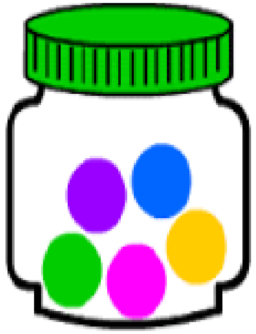
They are the opposite of independent events which do not affect or are not affected by another event or events.

P(A and B) = P(A) x P(B|A)

A jar contains 1 blue, 1 green, 1 pink, 1 purple and 1 yellow ball.

One ball is taken from the jar and is not replaced (Event A).

If another ball is taken out of the jar, what is the probability that the first ball is blue and the second ball is green?



Because the first ball is not replaced, the sample space of the second event (Event B) is changed.

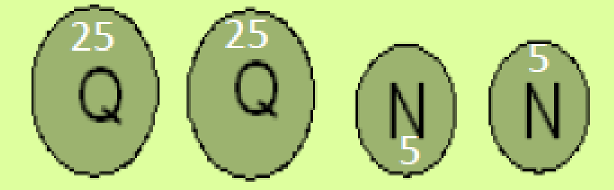
Event A (blue) sample space = 5 balls.
Event B (green) sample space = 4 balls.

P(blue then green) = P(blue) x P(green) = 1/5 x 1/4 = 1/20.

Independent and Dependent Events

Two events A and B are said to be **independent** if the fact that one event has occurred **does not affect** the probability that the other event will occur.

If whether or not one event occurs **does affect** the probability that the other event will occur, then the two events are said to be **dependent**.



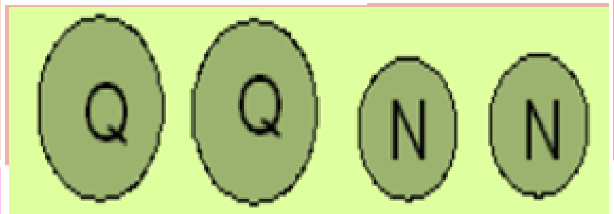
EXAMPLE:

A woman's pocket contains two quarters and two nickels.
She randomly extracts one of the coins and, after looking at it, replaces it before picking a second coin. Let Q1 be the event that the first coin is a quarter and Q2 be the event that the second coin is a quarter. Are Q1 and Q2 independent events?

Since the first coin that was selected is replaced, whether or not Q1 occurred has no effect on the probability that the second coin will be a quarter, P(Q2). In either case P(Q2) = 2/4 = 1/2 ,regardless of whether Q1 occurred.

EXAMPLE:

A woman's pocket contains two quarters and two nickels.
She randomly extracts one of the coins, and without placing it back into her pocket, she picks a second coin. let Q1 => the first coin is a quarter, and Q2 => the second coin is a quarter.
Are Q1 and Q2 independent events?



Q1 and Q2 are not independent. They are dependent.

Since the first coin that was selected is not replaced, whether Q1 occurred does affect the probability that the second coin is a quarter

If Q1 occurred , then P(Q2) = 1/3.
However, if Q1 has not occurred , then P(Q2) = 2/3

NOTE:

Disjoint events => Whether or not it is possible for the events to occur at the same time.
Independent events => Whether or not the occurrence of one event affects the probability of the occurrence of the other.

Independent Events

The outcome of one event **does not** affect the outcome of the other.

If A and B are independent events then the probability of both occurring is

$$P(A \text{ and } B) = P(A) \times P(B)$$

Dependent Events

The outcome of one event affects the outcome of the other.

If A and B are dependent events then the probability of both occurring is

$$P(A \text{ and } B) = P(A) \times P(B|A)$$

Probability of B given A

In Exercises 1 and 2, tell whether the events are independent or dependent. Explain your reasoning.

1. You toss a coin. Then you roll a number cube.

ANSWER

The coins toss does not affect the roll of a dice, so the events are independent.

2. You randomly choose 1 of 10 marbles. Then you randomly choose one of the remaining 9 marbles.

ANSWER

There is one fewer number in the bag for the second draw, so the events are dependent.

School Fair Your class is raising money by operating a ball toss game. You estimate that about 1 out of every 25 balls tossed results in a win. What is the probability that someone will win on two tosses in a row?

(A) $\frac{1}{625}$ (B) $\frac{1}{50}$ (C) $\frac{2}{25}$ (D) $\frac{25}{2}$

The tosses are independent events, because the outcome of a toss does not affect the probability of the next toss resulting in a win.

So the probability of each event is $\frac{1}{25}$.

$$P(\text{win and win}) = P(\text{win}) \cdot P(\text{win}) = \frac{1}{25} \cdot \frac{1}{25}$$

ANSWER

The probability of two winning tosses in a row is $\frac{1}{625}$.

The correct answer is A. (A) (B) (C) (D)

Multiplication Rule

If A and B are two INDEPENDENT events, then P(A and B) = P(A) * P(B).

When dealing with probability rules, the word "and" will always be associated with the operation of multiplication; hence the name of this rule, "The Multiplication Rule."

P(B after A) can also be written as P(B | A)

If A and B are DEPENDENT EVENTS, then the probability of A happening AND the probability of B happening, given A, is P(A) × P(B after A).

P(A and B) = P(A) × P(B after A)

then P(A and B) = P(A) × P(B | A)

This concludes that P(A | B) = $\frac{P(A \cap B)}{P(B)}$, where P(A | B) is the **conditional probability** of A when event B has already happened

Independent vs. Dependent Events

Using the bag of marbles on the left, what is the probability of pulling a white marble two times in a row? $P(white, white)$

When you put 1st marble back in (*Independent Events*)

With replacement

$$\frac{4}{10} * \frac{4}{10}$$

$$\frac{2}{5} * \frac{2}{5} = \frac{4}{25}$$

When you KEEP 1st marble (*Dependent Events*)

Without replacement

$$\frac{4}{10} * \frac{3}{9}$$

$$\frac{2}{5} * \frac{1}{3} = \frac{2}{15}$$

Independent vs. Dependent Events

Using the bag of marbles on the left, what is the probability of pulling a white marble, then a striped marble? $P(white, striped)$

When you put 1st marble back in (*Independent Events*)

With replacement

$$\frac{4}{10} * \frac{4}{10}$$

$$\frac{2}{5} * \frac{2}{5} = \frac{4}{25}$$

When you KEEP 1st marble (*Dependent Events*)

Without replacement

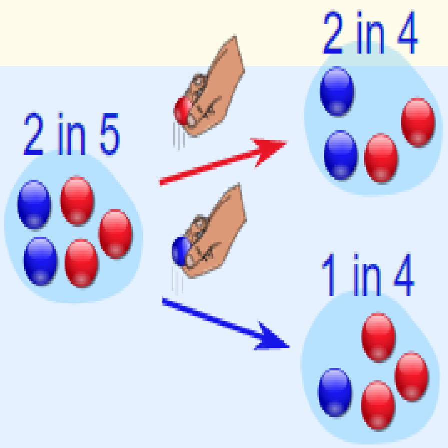
$$\frac{4}{10} * \frac{4}{9}$$

$$\frac{2}{5} * \frac{4}{9} = \frac{8}{45}$$

Conditional Probability

*How to handle **Dependent Events***

Example: Marbles in a Bag



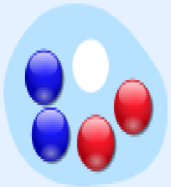
2 blue and 3 red marbles are in a bag.

What are the chances of getting a blue marble?

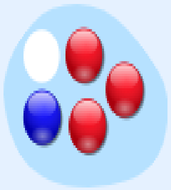
The chance is **2 in 5**

But after taking one out the chances change!

So the next time:



if we got a **red** marble before, then the chance of a blue marble next is **2 in 4**



if we got a **blue** marble before, then the chance of a blue marble next is **1 in 4**

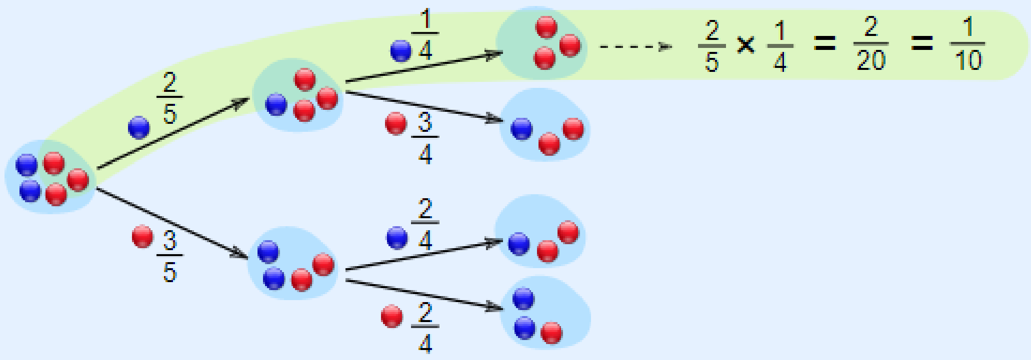
Tree Diagram

A Tree Diagram: is a wonderful way to picture what is going on, so let's build one for our marbles example.

There is a 2/5 chance of pulling out a Blue marble, and a 3/5 chance for Red:

Now we can answer questions like **"What are the chances of drawing 2 blue marbles?"**

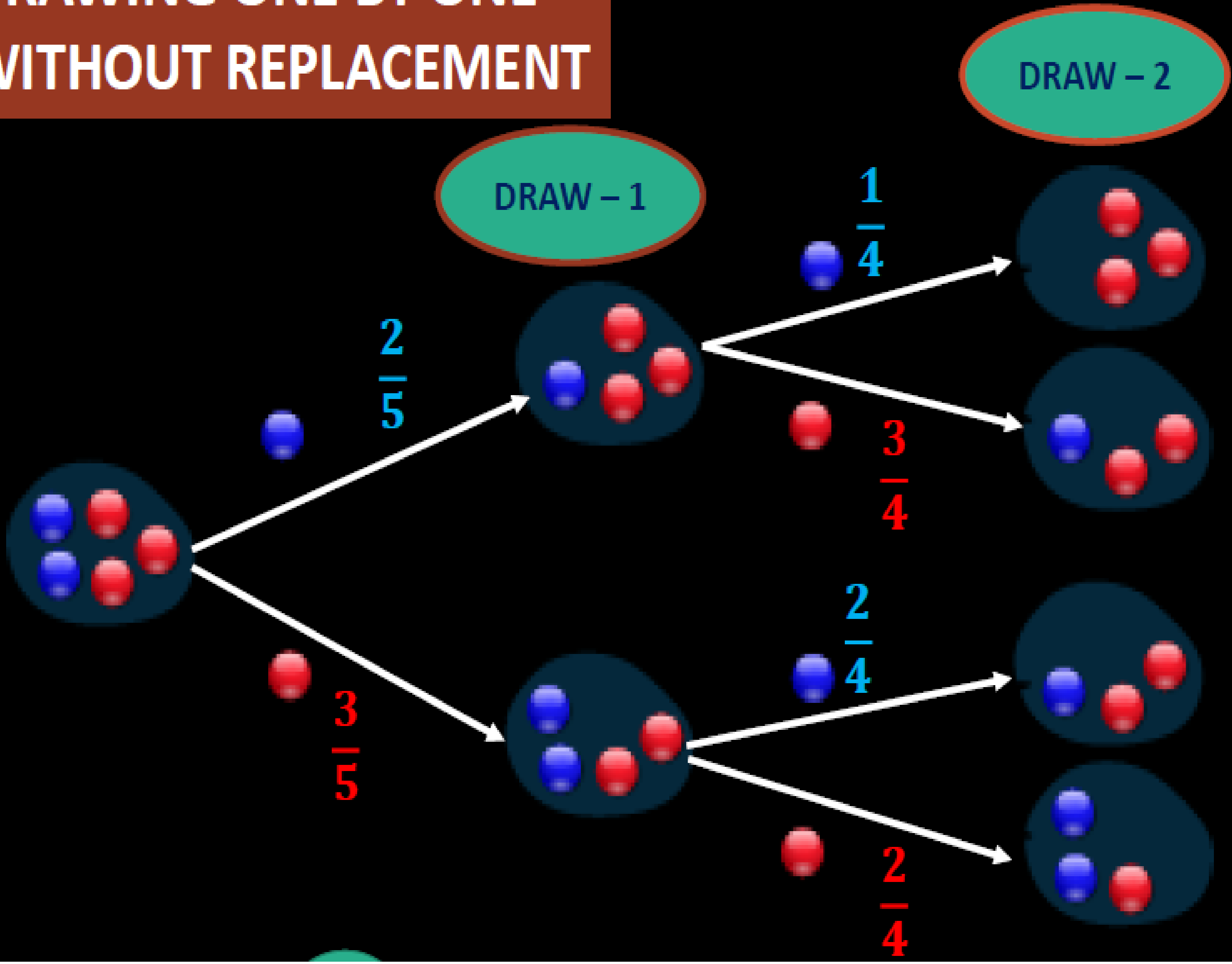
Answer: it is a **2/5 chance** followed by a **1/4 chance**:



Did you see how we multiplied the chances? And got 1/10 as a result.

The chances of drawing 2 blue marbles is 1/10

DRAWING ONE BY ONE WITHOUT REPLACEMENT



$P(BB) = \frac{2}{5} \times \frac{1}{4} = \frac{2}{20}$

$P(BR) = \frac{2}{5} \times \frac{3}{4} = \frac{6}{20}$

$P(RB) = \frac{3}{5} \times \frac{2}{4} = \frac{6}{20}$

$P(RR) = \frac{3}{5} \times \frac{2}{4} = \frac{6}{20}$

Conditional Probability Notation

Conditional probability of A given B is denoted by:

$$P(A \mid B)$$

PROBABILITY OF A GIVEN B

Conditional Probability Formula

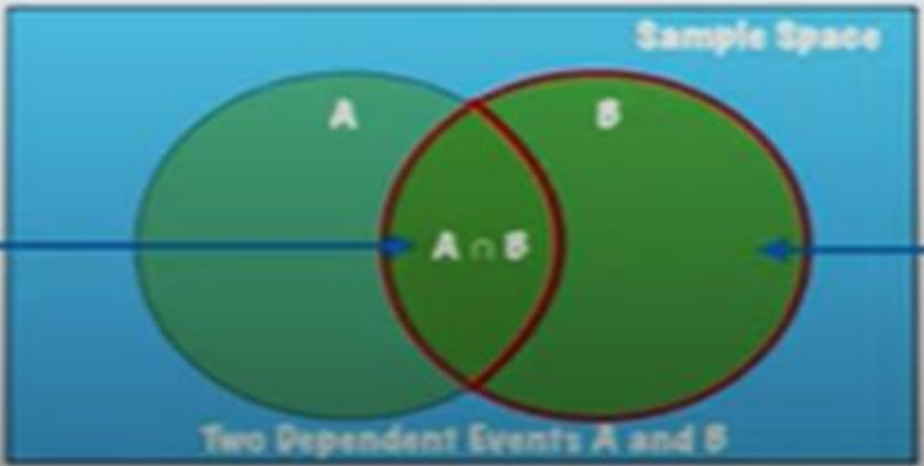
$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

For $P(B) > 0$

If $P(A \mid B) = 0$, then event A and event B are mutually exclusive events

Venn Diagram

$P(A \cap B)$



$P(B)$

Properties of conditional probability

Let E and F be events of a sample space S of an experiment, then we have

Property 1 $P(S|F) = P(F|F) = 1$

We know that

$$P(S|F) = \frac{P(S \cap F)}{P(F)} = \frac{P(F)}{P(F)} = 1$$

Also $$P(F|F) = \frac{P(F \cap F)}{P(F)} = \frac{P(F)}{P(F)} = 1$$

Thus $$P(S|F) = P(F|F) = 1$$

Property 2 *If A and B are any two events of a sample space S and F is an event of S such that $P(F) \neq 0$, then*

$$P((A \cup B)|F) = P(A|F) + P(B|F) - P((A \cap B)|F)$$

When A and B are disjoint events, then

$$P((A \cap B)|F) = 0$$

$$\Rightarrow \quad P((A \cup B)|F) = P(A|F) + P(B|F)$$

Property 3 $P(E'|F) = 1 - P(E|F)$

From Property 1, we know that $P(S|F) = 1$

$$\Rightarrow \quad P(E \cup E'|F) = 1 \qquad \text{since } S = E \cup E'$$

$$\Rightarrow \quad P(E|F) + P(E'|F) = 1 \qquad \text{since E and E}' \text{ are disjoint events}$$

Thus, $$P(E'|F) = 1 - P(E|F)$$

Example 1 If $P(A) = \frac{7}{13}$, $P(B) = \frac{9}{13}$ and $P(A \cap B) = \frac{4}{13}$, evaluate $P(A|B)$.

Solution We have $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{4}{13}}{\frac{9}{13}} = \dfrac{4}{9}$

Example 2 A family has two children. What is the probability that both the children are boys given that at least one of them is a boy ?

Solution Let b stand for boy and g for girl. The sample space of the experiment is

$$S = \{(b,\ b),\ (g,\ b),\ (b,\ g),\ (g,\ g)\}$$

Let E and F denote the following events :

E : 'both the children are boys'

F : 'at least one of the child is a boy'

Then $\quad E = \{(b,b)\}$ and $F = \{(b,b),\ (g,b),\ (b,g)\}$

Now $\quad E \cap F = \{(b,b)\}$

Thus $\quad P(F) = \dfrac{3}{4}$ and $P(E \cap F) = \dfrac{1}{4}$

Therefore $\quad P(E|F) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{\frac{1}{4}}{\frac{3}{4}} = \dfrac{1}{3}$

Example 5 A die is thrown three times. Events A and B are defined as below:

A : 4 on the third throw

B : 6 on the first and 5 on the second throw

Find the probability of A given that B has already occurred.

Solution The sample space has 216 outcomes.

Now $$A = \begin{Bmatrix} (1,1,4) & (1,2,4) & \dots & (1,6,4) & (2,1,4) & (2,2,4) & \dots & (2,6,4) \\ (3,1,4) & (3,2,4) & \dots & (3,6,4) & (4,1,4) & (4,2,4) & \dots & (4,6,4) \\ (5,1,4) & (5,2,4) & \dots & (5,6,4) & (6,1,4) & (6,2,4) & \dots & (6,6,4) \end{Bmatrix}$$

$$B = \{(6,5,1), (6,5,2), (6,5,3), (6,5,4), (6,5,5), (6,5,6)\}$$

and $$A \cap B = \{(6,5,4)\}.$$

Now $$P(B) = \frac{6}{216} \text{ and } P(A \cap B) = \frac{1}{216}$$

Then $$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{216}}{\frac{6}{216}} = \frac{1}{6}$$

Question 1:

Given that E and F are events such that

$P(E)=0.6,\ P(F)=0.3$ and, find $P(E|F)$ and $P(F|E)$.

Solution 1:

It is given that $P(E)=0.6,\ P(F)=0.3$, and $P(E\cap F)=0.2$

$$\Rightarrow P(E|F)=\frac{P(E\cap F)}{P(F)}=\frac{0.2}{0.3}=\frac{2}{3}$$

$$\Rightarrow P(F|E)=\frac{P(E\cap F)}{P(F)}=\frac{0.2}{0.6}=\frac{1}{3}$$

Question 3:

If $P(A)=0.8,\ P(B)=0.5$ and $P(B|A)=0.4$, find

(i) $P(A\cap B)$

(ii) $P(A|B)$

(iii) $P(A\cup B)$

Question 4:

Evaluate $P(A\cup B)$, if $2P(A)=P(B)=\frac{5}{13}$ and $P(A|B)=\frac{2}{5}$

Solution 4:

It is given that, $2P(A)=P(B)=\frac{5}{13}$

$$\Rightarrow P(A)=\frac{5}{26}\ and P(B)=\frac{5}{13}$$

$$P(A|B)=\frac{2}{5}$$

$$\Rightarrow \frac{P(A\cap B)}{P(B)}=\frac{2}{5}$$

$$\Rightarrow P(A\cap B)=\frac{2}{5}\times p(B)=\frac{2}{5}\times\frac{5}{13}=\frac{2}{13}$$

It is known that, $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$$\Rightarrow P(A\cup B)=\frac{5}{26}+\frac{5}{13}-\frac{2}{13}$$

$$\Rightarrow P(A\cup B)=\frac{5+10-4}{26}$$

$$\Rightarrow P(A\cup B)=\frac{11}{26}$$

Class Work Questions from Ex:13.1

7. Two coins are tossed once, where

(i) E : tail appears on one coin, F : one coin shows head

(ii) E : no tail appears, F : no head appears

9. Mother, father and son line up at random for a family picture

E : son on one end, F : father in middle

13. An instructor has a question bank consisting of 300 easy True / False questions, 200 difficult True / False questions, 500 easy multiple choice questions and 400 difficult multiple choice questions. If a question is selected at random from the question bank, what is the probability that it will be an easy question given that it is a multiple choice question?

14. Given that the two numbers appearing on throwing two dice are different. Find the probability of the event 'the sum of numbers on the dice is 4'.

Determine P(E|F)

Question 7:

Two coins are tossed once, where

(i) E: tail appears on one coin, F: one coin shows head

(ii) E: not tail appears, F: no head appears

Solution 7:

If two coins are tossed once, then the sample space S is

$S = \{HH, HT, TH, TT\}$

(i) $E = \{HT, TH\}$

$F = \{HT, TH\}$

$\therefore E \cap F = \{HT, TH\}$

$$P(F) = \frac{2}{8} = \frac{1}{4}$$

$$P(E \cap F) = \frac{2}{8} = \frac{1}{4}$$

$$\therefore P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{2}{2} = 1$$

(ii) $E = \{HH\}$

$F = \{TT\}$

$\therefore E \cap F = \phi$

$P(F) = 1$ and $P(E \cap F) = 0$

$$\therefore P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{0}{1} = 0$$

Determine P(E|F)

Question 9:

Mother, father and son line up at random for a family picture

E: son on one end, F: father in middle

Solution 9:

If mother (M), father (F), and son (S) line up for the family picture, then the sample space will be

S = {MFS, MSF, FMS, FSM, SMF, SFM}

$\Rightarrow$ E = {MFS, FMS, SMF, SFM}

F = {MFS, SFM}

$\therefore E \cap F$ = {MFS, SFM}

$$P(E \cap F) = \frac{2}{6} = \frac{1}{3}$$

$$P(F) = \frac{2}{6} = \frac{1}{3}$$

$$\therefore P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{1}{3}}{\frac{1}{3}} = 1$$

Question 13:

An instructor has a question bank consisting of 300 easy True/False questions, 200 difficult True/False questions, 500 easy multiple choice questions and 400 difficult multiple choice questions. If a question is selected at random from the question bank, what is the probability that it will be an easy question given that it is a multiple choice question?

Solution 13:

The given data can be tabulated as

	True / False	Multiple choice	Total
Easy	300	500	800
Difficult	200	400	600
Total	500	900	1400

Let us denote E = easy questions, M = multiple choice questions, D = difficult questions, and T = True/False questions

Total number of questions $= 1400$

Total number of multiple choice questions $= 900$

Therefore, probability of selecting an essay multiple choice question is

$$P(E \cap M) = \frac{500}{1400} = \frac{5}{14}$$

Probability of selecting a multiple choice question, P(M), is $\frac{900}{1400} = \frac{9}{14}$

$P\,(E|M)$ Represents the probability that a random selected question will be an easy question, given that it is a multiple choice question.

$$\therefore P(E|M) = \frac{P(E \cap M)}{P(M)} = \frac{\frac{5}{14}}{\frac{9}{14}} = \frac{5}{9}$$

Therefore, the required probability is $\frac{5}{9}$.

Question 14:

Given that the two numbers appearing on throwing the two dice are different. Find the probability of the event 'the sum of numbers on the dice is 4'.

Solution 14:

When dice is thrown, number of observations in the sample space $= 6 \times 6 = 36$

Let A be the event that the sum of the numbers on the dice is 4 and B be the event that the two numbers appearing on throwing the two dice are different.

$\therefore A = \{(1, 3), (2, 2), (3, 1)\}$

$$B = \begin{Bmatrix} (1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{Bmatrix}$$

$A \cap B = \{(1,3),(3,1)\}$

$\therefore P(B) = \frac{30}{36} = \frac{5}{6}$ and $P(A \cap B) = \frac{2}{36} = \frac{1}{18}$

Let $P(A|B)$ represents the probability that the sum of the numbers on the dice is 4, given that the two numbers appearing on throwing the two dice are different.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{18}}{\frac{5}{6}} = \frac{1}{15}$$

Multiple Choice

A school library classifies its books as hardback or paperback, fiction or nonfiction, and illustrated or nonillustrated. Use the table at the right for Exercises 25–27.

		Illustrated	Non-illustrated
Hardback	Fiction	420	780
	Nonfiction	590	250
Paperback	Fiction	150	430
	Nonfiction	110	880

1. What is the probability that a book selected at random is a paperback, given that it is illustrated?

A. $\frac{260}{3610}$ B. $\frac{150}{1270}$ C. $\frac{260}{1270}$ D. $\frac{110}{150}$

2. What is the probability that a book selected at random is nonfiction, given that it is a nonillustrated hardback?

F. $\frac{250}{2040}$ G. $\frac{780}{1030}$ H. $\frac{250}{1030}$ I. $\frac{250}{780}$

3. What is the probability that a book selected at random is a paperback?

A. $\frac{1}{1570}$ B. $\frac{260}{1310}$ C. $\frac{1570}{2040}$ D. $\frac{1570}{3610}$

Home Work Questions from Ex:13.1

10. A black and a red dice are rolled.

    (a) Find the conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5.

    (b) Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

11. A fair die is rolled. Consider events $E = \{1,3,5\}$, $F = \{2,3\}$ and $G = \{2,3,4,5\}$

    Find

    (i) $P(E|F)$ and $P(F|E)$ (ii) $P(E|G)$ and $P(G|E)$

    (iii) $P((E \cup F)|G)$ and $P((E \cap F)|G)$

12. Assume that each born child is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls given that (i) the youngest is a girl, (ii) at least one is a girl?

15. Consider the experiment of throwing a die, if a multiple of 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 3'.

In each of the Exercises 16 and 17 choose the correct answer:

16. If $P(A) = \frac{1}{2}$, $P(B) = 0$, then $P(A|B)$ is

    (A) 0 (B) $\frac{1}{2}$

    (C) not defined (D) 1

17. If A and B are events such that $P(A|B) = P(B|A)$, then

    (A) $A \subset B$ but $A \neq B$ (B) $A = B$

    (C) $A \cap B = \phi$ (D) $P(A) = P(B)$

RECAP

Multiplication Rule of probability

$P(E \cap F) = P(E)\ P(F|E)$ **provided P(E) not equals to 0**

$= P(F)\ P(E|F)$ provided $P(\mathbf{F}) \neq 0$

Multiplication rule of probability for more than two events If E, F and G are three events of sample space, we have

$$P(E \cap F \cap G) = P(E)\ P(F|E)\ P(G|(E \cap F)) = P(E)\ P(F|E)\ P(G|EF)$$

INDEPENDENT EVENTS

Thus, E and F are two events such that the probability of occurrence of one of them is not affected by occurrence of the other.

Such events are called *independent events*.

Definition : Let E and F be two events associated with the same random experiment, then E and F are said to be independent if

$$P(E \cap F) = P(E)\ .\ P(F)$$

Three events A, B and C are said to be mutually independent, if

$$P(A \cap B) = P(A)\ P(B)$$

$$P(A \cap C) = P(A)\ P(C)$$

$$P(B \cap C) = P(B)\ P(C)$$

and $$P(A \cap B \cap C) = P(A)\ P(B)\ P(C)$$

If at least one of the above is not true for three given events, we say that the events are not independent.

Example 10 A die is thrown. If E is the event 'the number appearing is a multiple of 3' and F be the event 'the number appearing is even' then find whether E and F are independent ?

Solution We know that the sample space is $S = \{1, 2, 3, 4, 5, 6\}$

Now $\quad E = \{3, 6\}, F = \{2, 4, 6\}$ and $E \cap F = \{6\}$

Then $\quad P(E) = \frac{2}{6} = \frac{1}{3}, \; P(F) = \frac{3}{6} = \frac{1}{2}$ and $P(E \cap F) = \frac{1}{6}$

Clearly $\quad P(E \cap F) = P(E).\,P(F)$

Hence $\quad$ E and F are independent events.

Example 14 If A and B are two independent events, then the probability of occurrence of at least one of A and B is given by $1 - P(A') P(B')$

Solution We have

$$
\begin{aligned}
P(\text{at least one of A and B}) &= P(A \cup B) \\
&= P(A) + P(B) - P(A \cap B) \\
&= P(A) + P(B) - P(A)\,P(B) \\
&= P(A) + P(B)\,[1 - P(A)] \\
&= P(A) + P(B).\,P(A') \\
&= 1 - P(A') + P(B)\,P(A') \\
&= 1 - P(A')\,[1 - P(B)] \\
&= 1 - P(A')\,P(B')
\end{aligned}
$$

Example 12 Three coins are tossed simultaneously. Consider the event E 'three heads or three tails', F 'at least two heads' and G 'at most two heads'. Of the pairs (E,F), (E,G) and (F,G), which are independent? which are dependent?

Solution The sample space of the experiment is given by

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Clearly $E = \{HHH, TTT\}$, $F = \{HHH, HHT, HTH, THH\}$

and $G = \{HHT, HTH, THH, HTT, THT, TTH, TTT\}$

Also $E \cap F = \{HHH\}$, $E \cap G = \{TTT\}$, $F \cap G = \{HHT, HTH, THH\}$

Therefore $P(E) = \frac{2}{8} = \frac{1}{4}$, $P(F) = \frac{4}{8} = \frac{1}{2}$, $P(G) = \frac{7}{8}$

and $P(E \cap F) = \frac{1}{8}$, $P(E \cap G) = \frac{1}{8}$, $P(F \cap G) = \frac{3}{8}$

Also $P(E) \,.\, P(F) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$, $P(E) \cdot P(G) = \frac{1}{4} \times \frac{7}{8} = \frac{7}{32}$

and $P(F) \,.\, P(G) = \frac{1}{2} \times \frac{7}{8} = \frac{7}{16}$

Thus $P(E \cap F) = P(E) \,.\, P(F)$

$P(E \cap G) \neq P(E) \,.\, P(G)$

and $P(F \cap G) \neq P(F) \,.\, P(G)$

Hence, the events (E and F) are independent, and the events (E and G) and (F and G) are dependent.

EX: 13.2

Question 2:

Two cards are drawn at random and without replacement from a pack of 52 playing cards. Finds the probability that both the cards are black.

Solution 2:

There are 26 black cards in a deck of 52 cards.

Let P(A) be the probability of getting a black card in the first draw.

$$\therefore P(A)=\frac{26}{52}=\frac{1}{2}$$

Let P(B) be the probability of getting a black card on second draw.

Since the card is not replaced,

$$\therefore P(B)=\frac{25}{51}$$

Thus, probability of getting both the cards black $=\frac{1}{2}\times\frac{25}{51}=\frac{25}{102}$

Question 3:

A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale, otherwise, it is rejected. Find the probability that a box containing 15 oranges out of which 12 are good and 3 are bad ones will be approved for sale.

Solution 3:

Let A, B, and C be the respective events that the first, second, and the third drawn orange is good.

Therefore, probability that first drawn orange is good, $P(A)=\frac{12}{15}$

The oranges are not replaced.

Therefore, probability of getting second orange good, $P(B)=\frac{11}{14}$

Similarly, probability of getting third orange good, $P(C)=\frac{10}{13}$

The box is approved for sale, if all the three oranges are good.

Thus, probability of getting all the oranges good $= =\frac{12}{15}\times\frac{11}{14}\times\frac{10}{13}=\frac{44}{91}$

Therefore, the probability that the box is approved for sale is $\frac{44}{91}$.

Question 7:

Given that the events A and B are such that $P(A)=\frac{1}{2}$, $P(A\cup B)=\frac{3}{5}$ and $\mathrm{P(B)=p}$. Find p if they are

(i) mutually exclusive

(ii) independent.

Solution 7:

It is given that $P(A)=\frac{1}{2}$, $P(A\cup B)=\frac{3}{5}$ and $\mathrm{P(B)=p}$

(i) When A and B are mutually exclusive, $A\cap B=\phi$

$\therefore P(A\cap B)=0$

It is known that, $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$$\Rightarrow \frac{3}{5}=\frac{1}{2}+p-0$$

$$\Rightarrow p=\frac{3}{5}-\frac{1}{2}=\frac{1}{10}$$

(ii) When A and B are independent, $P(A\cap B)=P(A).P(B)=\frac{1}{2}p$

It is known that, $P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$$\Rightarrow \frac{3}{5}=\frac{1}{2}+p-\frac{1}{2}p$$

$$\Rightarrow \frac{3}{5}=\frac{1}{2}+\frac{p}{2}$$

$$\Rightarrow \frac{p}{2}=\frac{3}{5}-\frac{1}{2}=\frac{1}{10}$$

$$\Rightarrow p=\frac{2}{10}=\frac{1}{5}$$

Question 9:

If A and B are two events such that $P(A)\frac{1}{4}$, $P(B)=\frac{1}{2}$ and $P(A\cap B)=\frac{1}{8}$, find P(not A and not B)

Solution 9:

It is given that, $P(A)\frac{1}{4}$ and $P(A\cap B)=\frac{1}{8}$

P (not on A and not on B) $= P(A'\cap B')$

P (not on A and not on B) $= P((A\cup B))'$ $\quad [A'\cap B' = (A\cup B)']$

$=1-P(A\cup B)$

$=1-[P(A)+P(B)-P(A\cap B)]$

$=1-\frac{5}{8}$

Question 12:

A die tossed thrice. Find the probability of getting an odd number at least once.

Solution 12:

Probability of getting an odd number in a single throw of a die $=\frac{3}{6}=\frac{1}{2}$

Similarly, probability of getting an even number $=\frac{3}{6}=\frac{1}{2}$

Probability of getting an even number three times $=\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$

Therefore, probability of getting an odd number at least once

= 1- probability of getting an odd number in none of the throws

= 1-probability of getting an even number thrice

$=1-\frac{1}{8}$

$=\frac{7}{8}$

Question 13:

Two balls are drawn at random with replacement from a box containing 10 black and 8 red balls. Find the probability that

(i) Both balls are red.

(ii) First ball is black and second is red.

(iii) One of them is black and other is red.

Solution 13:

Total number of balls = 18

Number of red balls = 8

Number of black balls = 10

(i) Probability of getting a red ball in the first draw $= \frac{8}{18} = \frac{4}{9}$

The ball is replaced after the first draw.

$\therefore$ Probability of getting a red ball in the second draw $= \frac{8}{18} = \frac{4}{9}$

Therefore, probability of getting both the balls red $= \frac{4}{9} \times \frac{4}{9} = \frac{16}{81}$

(ii) Probability of getting first ball black $= \frac{10}{18} = \frac{5}{9}$

The ball is replaced after the first draw.

Probability of getting second ball as red $= \frac{8}{18} = \frac{4}{9}$

Therefore, probability of getting first ball as black and second ball as red $= \frac{5}{9} \times \frac{4}{9} = \frac{20}{81}$

(iii) Probability of getting first ball as red $= \frac{8}{18} = \frac{4}{9}$

The ball is replaced after the first draw.

Probability of getting second ball as black $= \frac{10}{18} = \frac{5}{9}$

Therefore, probability of getting first ball as black and second ball as red $= \frac{4}{9} \times \frac{5}{9} = \frac{20}{81}$

Therefore, probability that one of them is black and other is red

= Probability of getting first ball black and second as red

+ Probability of getting first ball red and second ball black

$= \frac{20}{81} + \frac{20}{81}$

$= \frac{40}{81}$

Question 14:

Probability of solving specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively.

If both try to solve the problem independently, find the probability that

(i) the problem is solved

(ii) exactly one of them solves the problem

Solution 14:

Probability of solving the problem by A, $P(A)=\frac{1}{2}$,

Probability of solving the problem by B, $P(B)=\frac{1}{3}$

Since the problem is solved independently by A and B,

$\therefore P(AB)=P(A).P(B)=\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$

$P(A')=1-P(A)=1-\frac{1}{2}=\frac{1}{2}$

$P(B')=1-P(B)=1-\frac{1}{3}=\frac{2}{3}$

(i) Probability that the problem is solved $=P(A\cup B)$

$=P(A)+P(B)-P(AB)$

$=\frac{1}{2}+\frac{1}{3}-\frac{1}{6}$

$=\frac{4}{6}$

$=\frac{2}{3}$

(ii) Probability that exactly one of them solves the problem is given by,

$P(A).P(B')+P(B).P(A')$

$=\frac{1}{2}\times\frac{2}{3}+\frac{1}{2}\times\frac{1}{3}$

$=\frac{1}{3}+\frac{1}{6}$

$=\frac{1}{2}$

Question 16:

In a hostel, 60% of the students read Hindi newspaper, 40% read English newspaper and 20% read both Hindi and English newspapers. A student is selected at random.

(a) Find the probability that she reads neither Hindi nor English newspapers.

(b) If she reads Hindi newspaper, find the probability that she reads English newspaper.

(c) If she reads English newspaper, find the probability that she reads Hindi newspaper.

Solution 16:

Let H denote the students who read Hindi newspaper and E denote the students who read English newspaper.

It is given that,

$$P(H)=60\%=\frac{60}{100}=\frac{3}{5}$$

$$P(E)=40\%=\frac{40}{100}=\frac{2}{5}$$

$$P(H\cap E)=20\%=\frac{20}{100}=\frac{1}{5}$$

(i) Probability that a student reads Hindi and English newspaper is,

$$P(H\cup E)'=1-P(H\cup E)$$
$$=1-\{P(H)+P(E)-P(H\cap E)\}$$
$$=1-\left(\frac{3}{5}+\frac{2}{5}-\frac{1}{5}\right)$$
$$=1-\frac{4}{5}$$
$$=\frac{1}{5}$$

(ii) Probability that a randomly chosen student reads English newspaper, if she reads Hindi newspaper, is given by $P(E|H)$.

$$P(E|H)=\frac{P(E\cap H)}{P(H)}$$

$$=\frac{\frac{1}{5}}{\frac{3}{5}}$$

$$=\frac{1}{3}$$

(iii) Probability that a random chosen student reads Hindi newspaper, if she reads English newspaper, is given by $P(H|E)$.

$$P(H|E)=\frac{P(H\cap E)}{P(E)}$$

$$=\frac{\frac{1}{5}}{\frac{2}{5}}$$

$$=\frac{1}{2}$$

Home Work Questions from Ex:13.2

5. A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event, 'the number is even,' and B be the event, 'the number is red'. Are A and B independent?

6. Let E and F be events with $P(E) = \frac{3}{5}$, $P(F) = \frac{3}{10}$ and $P(E \cap F) = \frac{1}{5}$. Are E and F independent?

10. Events A and B are such that $P(A) = \frac{1}{2}$, $P(B) = \frac{7}{12}$ and P(not A or not B) $= \frac{1}{4}$. State whether A and B are independent ?

11. Given two independent events A and B such that $P(A) = 0.3$, $P(B) = 0.6$. Find

(i) P(A and B) (ii) P(A and not B)

(iii) P(A or B) (iv) P(neither A nor B)

Choose the correct answer in Exercises 17 and 18.

17. The probability of obtaining an even prime number on each die, when a pair of dice is rolled is

(A) 0 (B) $\frac{1}{3}$ (C) $\frac{1}{12}$ (D) $\frac{1}{36}$

18. Two events A and B will be independent, if

(A) A and B are mutually exclusive

(B) $P(A'B') = [1 - P(A)]\,[1 - P(B)]$

(C) $P(A) = P(B)$

(D) $P(A) + P(B) = 1$